ETH zürich

Institute for Dynamic Systems and Control

Control Systems 1

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Welcome!

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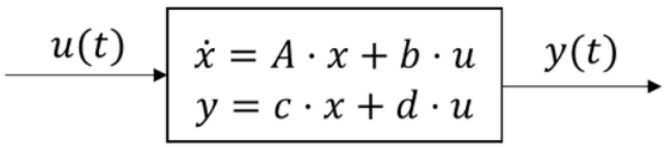
Today

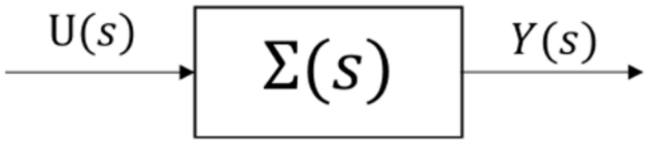
- **Extensive** Repetition Session 5
- Theory Recap
  - Transfer Function Notations
  - Interjection: Transfer Function Visualisations
  - Effect of Poles
  - Effect of Zeros
  - Pole Zero Cancellation
  - Initial and Final Value Theorem
- Q&A Session / Done

Repetition Session 5

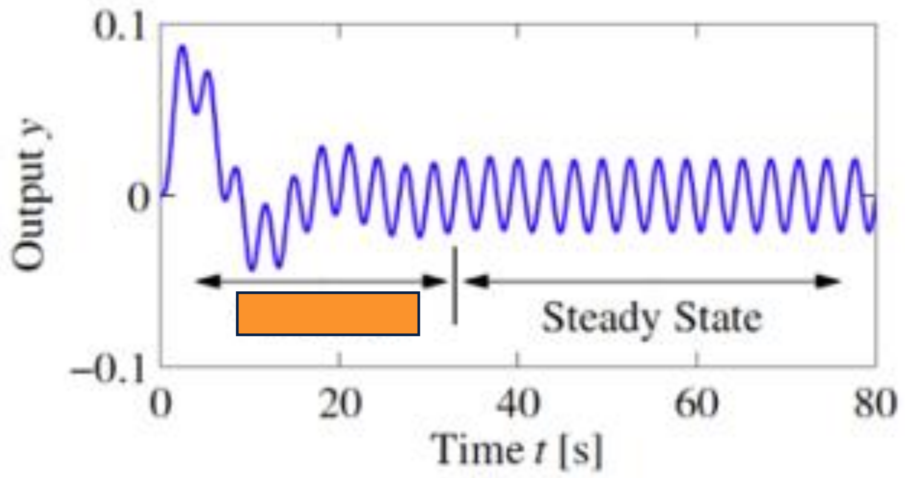
What we did

Time Domain → **Laplace Domain**





How do we call the first part of the response?



A) Decaying Response

C) Transient Response

B) First Response

D) Converging Response

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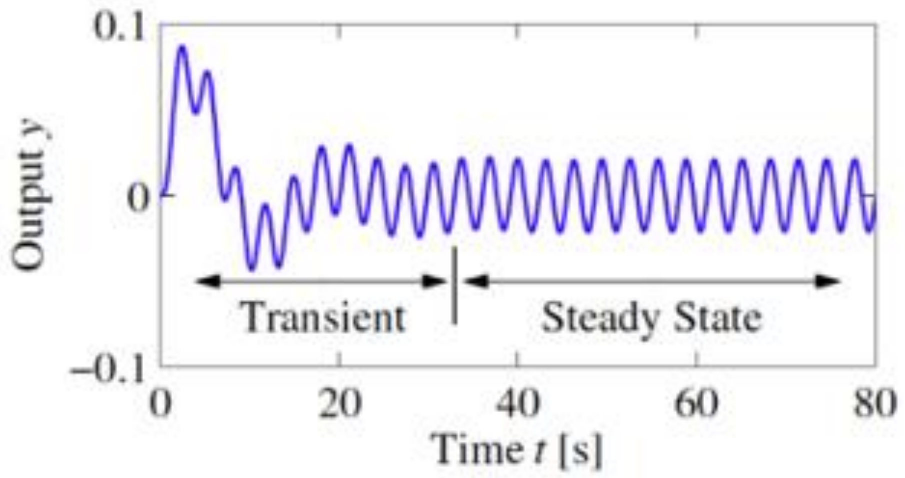

A) Decaying Response

C) Transient Response

B) First Response

D) Converging Response

What is the correct matrix A and C combination in the controlable canonical form to represent G(s)?

$$G(s) = \frac{s^2 + 2}{s^3 - 2s^2 - 5s + 6} + 10.$$

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & 5 & 2 \end{bmatrix}, \qquad C = \begin{bmatrix} 2 & 0 & 1 \end{bmatrix},$$

1.

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & 5 & 6 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 0 & 2 \end{bmatrix},$$

2.

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & 0 & -1 \end{bmatrix} \qquad C = \begin{bmatrix} -6 & 5 & 2 \end{bmatrix};$$

3.

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 6 & -5 & -2 \end{bmatrix} \qquad C = \begin{bmatrix} 2 & 0 & 1 \end{bmatrix},$$

4.

What is the correct matrix A and C combination in the controlable canonical form to represent G(s)?

$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix}$

$$G(s) = \frac{s^2 + 2}{s^3 - 2s^2 - 5s + 6} + 10.$$

$+a_2 \Rightarrow a_2 = -2$

$\Rightarrow -a_2 = 2$

$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & 5 & 2 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & 0 & 1 \end{bmatrix},$

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1.

2.

3.

4.

Expressing Complex (difficult) Inputs

Remember when we talked about wanting to express complex inputs as a linear combination of simpler more elementary ones? We can do that, since we are looking at **L(inear)TI SISO** systems.

$$\Sigma(\alpha u_1 + \beta u_2) = \alpha \Sigma u_1 + \beta \Sigma u_2, \quad \alpha, \beta \in \mathbb{R}.$$

$$u(t) = \alpha_1 u_1(t) + \alpha_2 u_2(t) + \cdots + \alpha_m u_m(t),$$

apply each input to the system separately sum the outputs

$$y(t) = \alpha_1 y_1(t) + \alpha_2 y_2(t) + \cdots + \alpha_m y_m(t).$$

Expressing Complex (difficult) Inputs

simple

Okay, so what did we do? We introduced an elementary input of the form $\boxed{u(t) = e^{st}}$

Here s is simply a complex variable of the form $\boxed{s = \sigma + j\omega}$ $= x + iy$
Later, we also called it our laplace variable.

Now we can construct a general input $\boldsymbol{u(t)}$ as a sum of different exponentials, weighted by $\boldsymbol{U_i}$

$$\boxed{u(t) = \sum_i U_i e^{s_i t},}$$

$i = 1$, $U_i = 1$, $u(t) = e^{st}$

Expressing Complex (difficult) Inputs

There is also a more general and **non discrete** (non countable) case, but telling us basically the same:

When working with the **inverse laplace transform.** We can write almost any function, as an infinite sum of weighted e^{st}-terms, weighted by the corresponding value of F(s).

Again, s is just a complex variable, that we also call laplace variable.

$U_i = [1, 5, 3, 4, \ldots]$

$U_i = F(s_i)$

$\sum_i$

$\frac{1}{2}e^{it} + \frac{1}{2}e^{-it} = \cos(t)$

$$f(t) = \cancel{\frac{1}{2\pi j}} \lim_{\omega\to\infty} \int_{\sigma-j\omega}^{\sigma+j\omega} \boxed{F(s)}\boxed{e^{st}}\,\mathrm{d}s.$$

Looks very similar to the form of the previous slide with: $u(t) = \sum_i \boxed{U_i}\boxed{e^{s_i t}},$

Expressing Complex (difficult) Inputs

When working with the Laplace Transform, we can write our **Input - Output relation** like this:
(derived in Lecture)

$$\boxed{Y(s) = G(s)U(s)}$$

Leading us to this:

$$\boxed{y(t) = \mathcal{L}^{-1}\{G(s)U(s)\}}$$

$$y_{ss}(t) = G(s) \cdot u(t)$$

Theory Recap

Transfer Function Notations

Transfer Function Notations

$\underbrace{\text{Transfer Function}}$

Let's call it «TF»

Numerator

On the right we have the general form, where we have the ratio of two polynomials + «feedthrough» d:

$$\boxed{G(s) = \frac{N(s)}{D(s)} = \frac{c_{n-1}s^{n-1} + c_{n-2}s^{n-2} + \cdots + c_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_0} + d.}$$

Denominator

There are however some more notations, that can come in quite handy, depending on our goal:

Partial Fraction Expansion

$$\boxed{G(s) = \frac{r_1}{s - p_1} + \frac{r_2}{s - p_2} + \cdots + \frac{r_n}{s - p_n} + d,}$$

Root Locus Form (next week)

$$\boxed{G(s) = \frac{k_{rl}}{s^q} \frac{(s - z_1)(s - z_2)\cdots(s - z_m)}{(s - p_1)(s - p_2)\cdots(s - p_{n-q})},}$$

Bode Form (in 3 weeks)

$$\boxed{G(s) = \frac{k_{\text{Bode}}}{s^q} \frac{\left(\frac{s}{-z_1} + 1\right)\left(\frac{s}{-z_2} + 1\right)\cdots\left(\frac{s}{-z_m} + 1\right)}{\left(\frac{s}{-p_1} + 1\right)\left(\frac{s}{-p_2} + 1\right)\cdots\left(\frac{s}{-p_{n-q}} + 1\right)}}$$

Partial Fraction Expansion

$\lim_{s \to p_1}$

$$\boxed{G(s) = \frac{r_1}{s-p_1} + \frac{r_2}{s-p_2} + \cdots + \frac{r_n}{s-p_n} + d,} \cdot (s-p_1)$$

As usual, **p** stands for the poles in our system. The **r**'s are the so called **residues**. They determine how much a certain pole is present in our system.

How to find~~e~~ the residues? We could use the actual **partial fraction expansion (Partialbruchzerlegung),** or make our life easier and use the so called **cover up method**. The given formula however, only works for non repeating poles.

$$\boxed{r_i = \lim_{s \to p_i} (s-p_i)G(s).}$$

r_1:

This would be the formula for order m repeating poles.

$$\boxed{r_i = \frac{1}{(m-1)!} \lim_{s \to p_i} \frac{\mathrm{d}^{m-1}}{\mathrm{d}s^{m-1}} \left((s-p_i)^m G(s)\right).}$$

Root Locus and Bode Form

Both of them are going to be important in the upcoming weeks. They show us in a different way where the poles and where the zeros of the system are. Additionaly they have a gain-factor k, and the integrator (pole at = 0) part infront.

$$G(s) = \frac{k_{rl}}{s^q} \frac{(s - z_1)(s - z_2) \cdots (s - z_m)}{(s - p_1)(s - p_2) \cdots (s - p_{n-q})},$$

$$G(s) = \frac{k_{\text{Bode}}}{s^q} \frac{\left(\frac{s}{-z_1} + 1\right)\left(\frac{s}{-z_2} + 1\right) \cdots \left(\frac{s}{-z_m} + 1\right)}{\left(\frac{s}{-p_1} + 1\right)\left(\frac{s}{-p_2} + 1\right) \cdots \left(\frac{s}{-p_{n-q}} + 1\right)}.$$

$(s-p) = s$

since $p=0$

$q =$ power of s, $\Rightarrow$ we have q integrators

Root Locus to Bode Form:

$$(s - z_i) \Rightarrow z_i \cdot \left(\frac{s}{-z_i} + 1\right)$$
$$(s - p_j) \Rightarrow p_j \cdot \left(\frac{s}{-p_j} + 1\right)$$

$$k_{\text{Bode}} = k_{\text{rl}} \cdot \frac{\prod_i z_i}{\prod_j p_j}$$

can have 0 integrators!

See how when we have a pole at zero (meaning p = 0), we can not just divide by p. This is why we have the extra integrator term infront

Transfer Functions Important Note

Root Locus

$$G(s) = \boxed{\frac{N(s)}{D(s)}} = \frac{c_{n-1}s^{n-1} + c_{n-2}s^{n-2} + \cdots + c_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_0} + d. = \boxed{\frac{k_{rl}}{s^q} \frac{(s - z_1)(s - z_2)\cdots(s - z_m)}{(s - p_1)(s - p_2)\cdots(s - p_{n-q})}},$$

first zero...

first pole ...

- Zeros = Roots of Numerator
- Poles = Roots of Denominator

See how in the Root Locus we have all the roots of N and D in **factorized** notation.

Generally for LTI SISO systems, de degree of the **Denominator** is **higher** than the **Numerator**.
Also, we often times say **d = 0** for simplicity.

Example

$$G(s) = \frac{s^2 + 4s - 5}{s^3 + 3s^2 + 2s}$$

1. Determine the poles and the zeros of the system

$$D(s) = s \cdot (\underbrace{s^2}_{a} + \underbrace{3s}_{b} + \underbrace{2}_{c})$$

$$p_0 = 0$$

$$p_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \Rightarrow \frac{-3 \pm \sqrt{9 - 8}}{2} = \frac{-3 \pm 1}{2}$$

$$\Rightarrow p_1 = -1, \quad p_2 = -2$$

Example

$$G(s) = \frac{s^2 + 4s - 5}{s^3 + 3s^2 + 2s}$$

1. Determine the poles and the zeros of the system

$$p_0 = 0,\ p_1 = -1,\ p_2 = -2.$$
$$z_0 = -5,\ z_1 = 1.$$

2. Represent G(s) in root locus form

$$\frac{(s-(-5))\cdot(s-(1))}{(s-0)\cdot(s-(-1))\cdot(s-(-2))} = \frac{1}{s}\,\frac{(s+5)\cdot(s-1)}{(s+1)\cdot(s+2)}$$

Example

$$G(s) = \frac{s^2 + 4s - 5}{s^3 + 3s^2 + 2s}$$

1. Determine the poles and the zeros of the system

$$p_0 = 0,\ p_1 = -1,\ p_2 = -2.$$
$$z_0 = -5,\ z_1 = 1.$$

2. Represent G(s) in root locus form

$$G(s) = \frac{(s+5)(s-1)}{s(s+2)(s+1)}$$

3. Represent G(s) in bode form

$$\frac{\left(\frac{s}{5}+1\right)\cdot\left(\frac{s}{-1}+1\right)}{s\cdot\left(\frac{s}{2}+1\right)\cdot\left(\frac{s}{1}+1\right)} \cdot \underbrace{\frac{5\cdot(-1)}{2\cdot 1}}_{-\frac{5}{2} = k_{Bode}}$$

Example

$$G(s) = \frac{s^2 + 4s - 5}{s^3 + 3s^2 + 2s}$$

$$r_1 = \lim_{s \to 0} \frac{\cancel{s} \cdot \overset{0}{(s+5)} \cdot \overset{0}{(s-1)}}{\cancel{s} \cdot \underbrace{\underset{0}{(s+2)}\underset{0}{(s+1)}}}$$

$$\Rightarrow \frac{5 \cdot (-1)}{2 \cdot 1} = -\frac{5}{2}$$

1. Determine the poles and the zeros of the system

$$p_0 = 0,\ p_1 = -1,\ p_2 = -2.$$
$$z_0 = -5,\ z_1 = 1.$$

2. Represent G(s) in root locus form

$$G(s) = \frac{(s+5)(s-1)}{s(s+2)(s+1)}$$

$$\frac{-\frac{5}{2}}{s} + \frac{\dots}{s+2} + \frac{\dots}{s+1}$$

3. Represent G(s) in bode form

$$G(s) = \frac{-\frac{5}{2}(\frac{s}{5}+1)(\frac{s}{-1}+1)}{s(\frac{s}{2}+1)(\frac{s}{1}+1)}$$

4. Represent G(s) in the partial fraction expansion

$$G(s) = \frac{\frac{-5}{2}}{s} + \frac{8}{s+1} + \frac{-\frac{9}{2}}{s+2}$$

$$r_i = \lim_{s \to p_i} (s - p_i) G(s).$$

Transfer Function Visualisations

Motivation Example

We want to write $u(t) = \cos(\omega t)$, as an function of complex exponential functions:

$$u(t) = \cos(\omega t) = \sum_i U_i e^{s_i t}$$

Now we need to decompose the cosine. Remember how cos can be written:

$$\cos(\omega t) = \frac{1}{2} e^{j\omega t} + \frac{1}{2} e^{-j\omega t}.$$

So now we see that $U_{1,2} = \frac{1}{2}$ and $s_{1,2} = \pm j\omega.$

$\cdot u(t)$

Now we want to compute the steady state response $y_{\text{ss}}(t) = G(s)$ or more generally:

$$y_{\text{ss}}(t) = \sum_i U_i G(s_i) e^{s_i t}.$$

Quizz: How can we write the sinus in exponentials?

$$\frac{e^{j\omega t} - e^{-j\omega t}}{2i}$$

Motivation Example

Plugging in the values we get:

$$\boxed{y_{ss}(t) = \frac{1}{2}G(j\omega)e^{j\omega t} + \frac{1}{2}G(-j\omega)e^{-j\omega t}.}$$

Let's use the fact that $G(j\omega)$ is a complex function that can be decomposed into a magnitude and a phase (in polar form) .

$$\boxed{M = |G(j\omega)| \quad \text{and} \quad \varphi = \angle(G(j\omega))}$$

angle

$$|G(j\omega)| = |G(-j\omega)|$$

So now our response becomes:

$$y_{ss}(t) = \frac{1}{2}\underbrace{Me^{j\varphi}}_{G(j\omega)}e^{j\omega t} + \frac{1}{2}\underbrace{Me^{-j\varphi}}_{G(-j\omega)}e^{-j\omega t} = M \cdot \underbrace{\left(\frac{1}{2}\dots\right)}_{\cos(\dots)}$$

Reformulating:

$$\boxed{y_{ss}(t) = M\cos(\omega t + \varphi)}$$

Motivation Example

$$u(t) = \cos(\omega t), \qquad \boxed{y_{ss}(t) = M\cos(\omega t + \varphi)}$$

We can conclude that for a sinusoidal input, the output is just another sinusoid with the **same frequency, shifted by a phase φ and multiplied by the magnitue M.**

We will later see how to compute the phase and magnitude of a transfer function.

Transfer Function Visualisation

Remember how the TF is just a complex number if a certain input is given?
Lets use this to see how the transfer function changes a certain input (similar example as before):

ω_0 ω ω_0 ω ω_0 ω

$$\boxed{u(t) = \sin(t) \qquad y_{ss}(t) = |G(j)|\sin(t + \angle G(j))} \longleftarrow s = j$$

So how do we actually determine the magnitude and phase of the transfer function?
Consider the factorized TF:

$$\boxed{G(s) = 2\frac{s+1}{(s+2)(s+1+j)(s+1-j)}}$$

We will look at two ways on how to determine magnitude an phase: **Computationally and Graphically**

Magnitude

X = poles

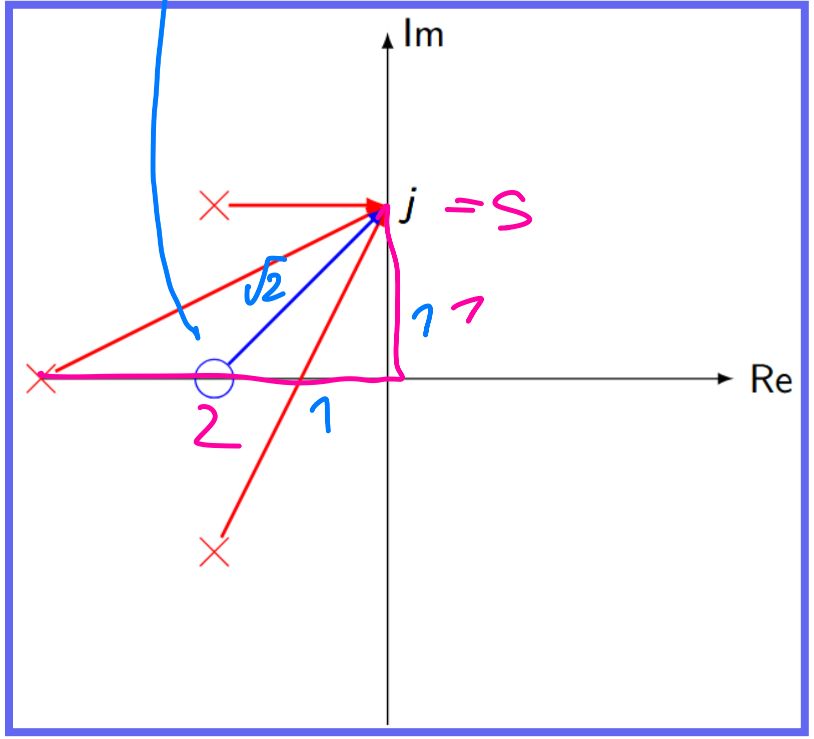
O = zeros

Generally for magnitudes: $|a \cdot b| = |a| \cdot |b|$

$z_1 = -1$

Our TF becomes: $$|G(s)| = 2\frac{|s+1|}{|s+2| \cdot |s+1+j| \cdot |s+1-j|}$$

Graphically speaking, |s - p| is just the length of the vector from p to s



Reading from the plot (knowing the position of poles and zeros)

$$|G(j)| = \boxed{2}\frac{\sqrt{2}}{\sqrt{5} \cdot \sqrt{5} \cdot 1} = \frac{\boxed{2}\sqrt{2}}{5}$$

$\sqrt{2^2+1^2} = \sqrt{5}$

Phase

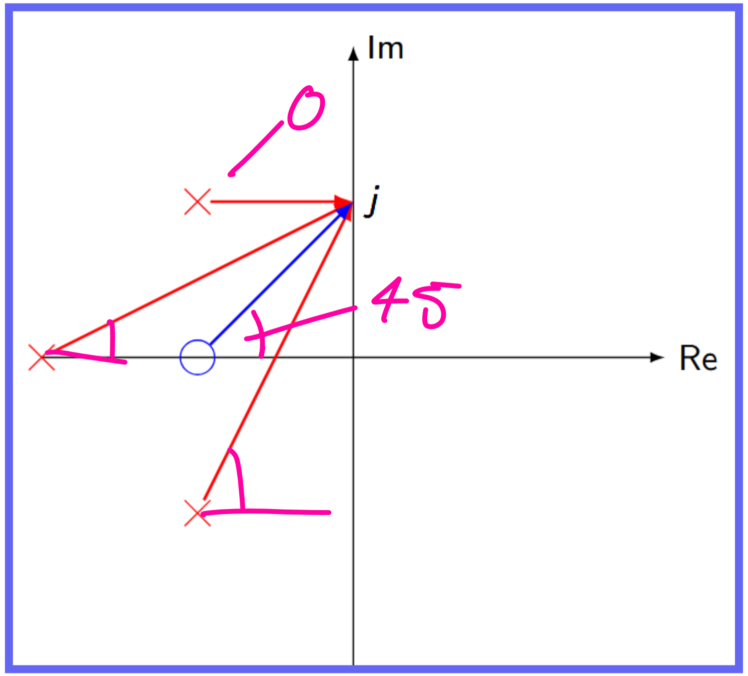
$\angle(x+iy) = \arctan\left(\frac{y}{x}\right)$

Generally for magnitudes: $\angle(a \cdot b) = (\angle a) + (\angle b)$

Our TF becomes:

$$\angle G(s) = \angle(2) + \angle(s+1) - \angle(s+2) - \angle(s+1+j) - \angle(s+1-j)$$

Graphically speaking $\angle(s-p)$ is just the angle formed by vector from p to s with the real axis

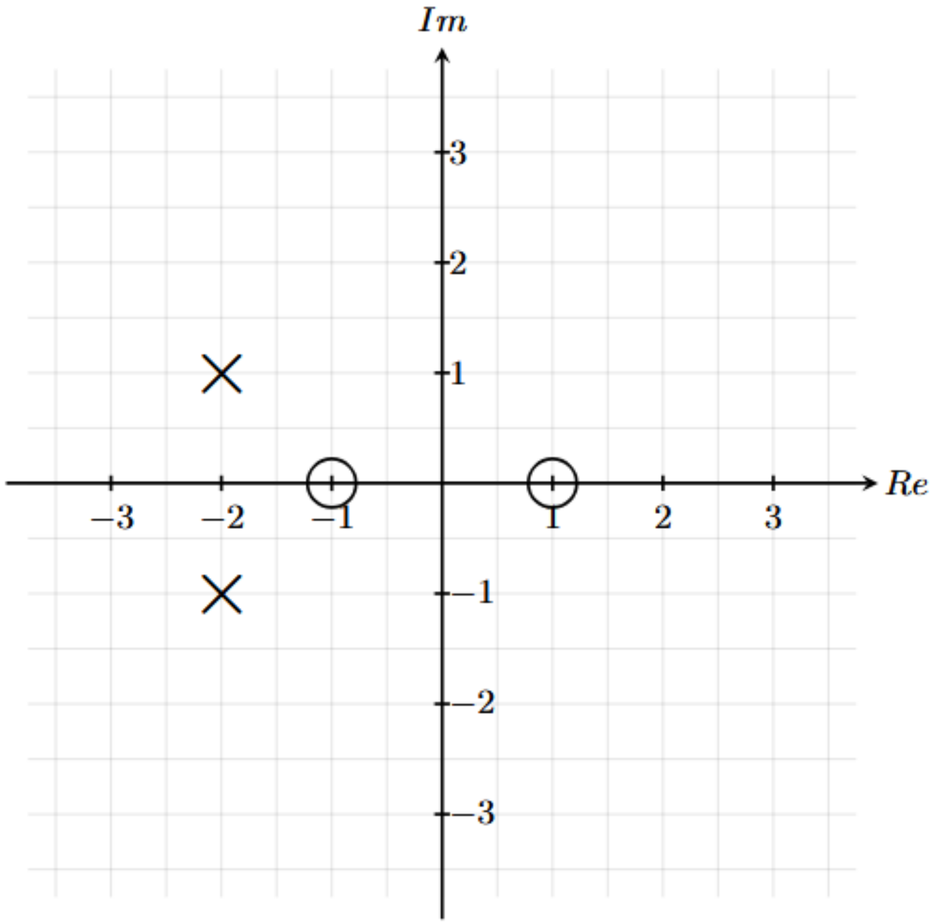


Reading from the plot (lowkey):

$$\angle G(j) = 0 + 45^\circ - \arctan(1/2) - \arctan(2) - 0^\circ = -45^\circ$$

$45 - \ldots 90 = -45^\circ$

HS 2023



What is the magnitude $|G(j)|$ of $G(s = j)$?

A) $= \frac{1}{2\sqrt{2}}$

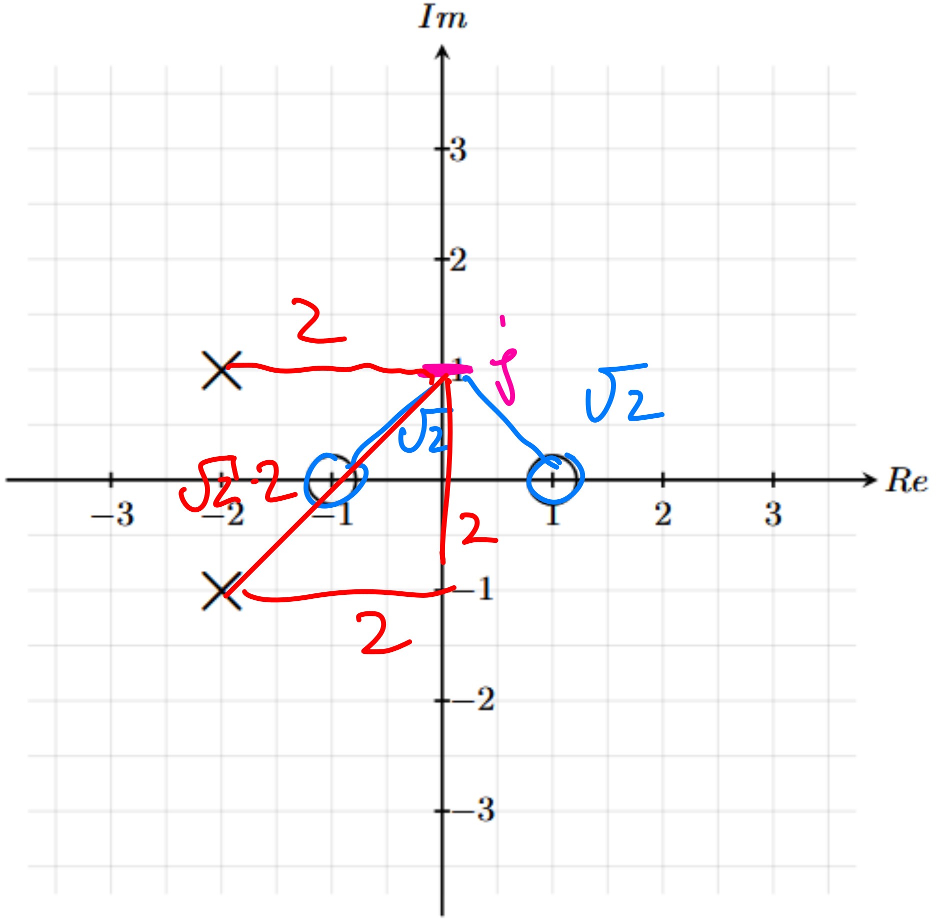
B) $= \frac{1}{4}$

C) $= \frac{1}{2}$

D) $= \frac{1}{\sqrt{2}}$

HS 2023

$$|G(j)| = \frac{\sqrt{2}\cdot\sqrt{2} = \cancel{2}}{\sqrt{2}\cdot 2\cdot\cancel{2}} = \frac{1}{2\sqrt{2}}$$



What is the magnitude $|G(j)|$ of $G(s = j)$?

A) $= \frac{1}{2\sqrt{2}}$

B) $= \frac{1}{4}$

C) $= \frac{1}{2}$

D) $= \frac{1}{\sqrt{2}}$

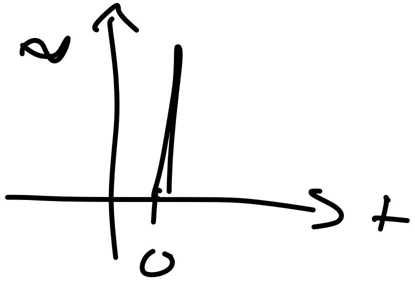
Effect of Poles

Effect of Poles

$f(t)$	$\mathscr{L}f(t) = F(s)$	$t^n \ (n = 0, 1, 2, \dots)$	$\frac{n!}{s^{n+1}}$
1	$\frac{1}{s}$	$\sin kt$	$\frac{k}{s^2+k^2}$
$e^{at}f(t)$	$F(s-a)$	$\cos kt$	$\frac{s}{s^2+k^2}$
$\mathcal{U}(t-a)$	$\frac{e^{-as}}{s}$	$\sin^2 kt$	$\frac{2k^2}{s(s^2+4k^2)}$
$f(t-a)\mathcal{U}(t-a)$	$e^{-as}F(s)$	$\cos^2 kt$	$\frac{s^2+2k^2}{s(s^2+4k^2)}$
$\delta(t)$	1	e^{at}	$\frac{1}{s-a}$
$\delta(t-a)$	e^{-sa}		
$\frac{d^n}{dt^n}\delta(t)$	s^n		

Let us consider how the system reacts to a dirac impulse input with zero initial condition.

$$x(0) = 0 \quad u(t) = \delta(t)$$



Looking at the s - domain answer and transforming the input, we get:

$$U(s) = 1. \quad \rightarrow \quad Y(s) = G(s) \cdot 1 = G(s)$$

Now we can already go back to the t - domain:

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\{G(s)\}$$

Effect of Poles

Now lets have a look at a transfer function in partial fraction expansion:

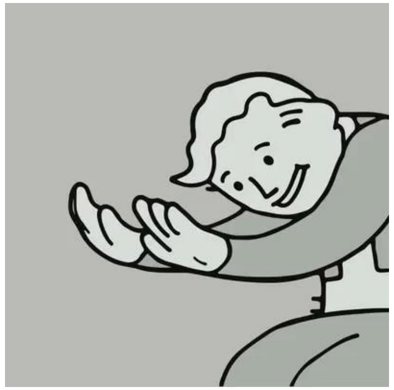
$$G(s) = \frac{r_1}{s - p_1} + \frac{r_2}{s - p_2} + \cdots + \frac{r_n}{s - p_n}$$

Luckily, in this form it is super easy to take the inverse laplace transform

$$\boxed{\mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\} = e^{at} \qquad \text{and} \qquad \mathcal{L}^{-1}\left\{\frac{r_i}{s-p_i}\right\} = r_i e^{p_i t}}$$

And now since the laplace transform is a linear transformation the output just becomes:

$$\boxed{y(t) = r_1 e^{p_1 t} + r_2 e^{p_2 t} + \cdots + r_n e^{p_n t}, \quad t \geq 0}$$

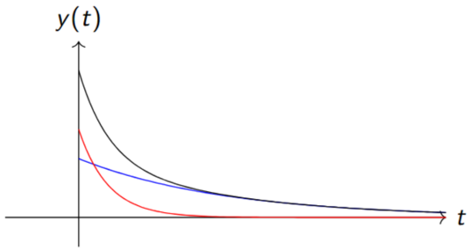


Hold up, this looks familiar…?

Effect of Poles

Example

$$y(t) = Ce^{At}x_0 = \begin{bmatrix} c_1 & c_2 \end{bmatrix} \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix} \begin{bmatrix} x_{0,1} \\ x_{0,2} \end{bmatrix} = c_1 e^{\lambda_1 t} x_{0,1} + c_2 e^{\lambda_2 t} x_{0,2}$$



Remember how the initial condition response, every eigenvalue of the matrix A was found in an exponential term and helped us to construct a solution?

Further, lets take look at the forced response for certain conditions:

$$D = 0,\; x(0) = 0,\; u(t) = \delta(t)$$

$$y(t) = \overset{0}{\cancel{Ce^{At}x_0}} + C\int_0^t e^{A(t-\tau)}\, B\, u(\tau)\, \mathrm{d}\tau + \overset{0}{\cancel{D\,u(t)}}.$$

$$y_{imp}(t) = C\int_0^t e^{A(t-\tau)}\, B\, \delta(\tau)\, \mathrm{d}\tau = \boxed{Ce^{At}B}$$

$$C \cdot e^{At} \cdot B$$

Effect of Poles

Let's apply the same input in the t - domain.

$$y(t) = Ce^{At}x_0 + C\int_0^t e^{A(t-\tau)}\,B\,u(\tau)\,\mathrm{d}\tau + D\,u(t).$$

with $D = 0,\ x(0) = 0,\ u(t) = \delta(t)$

$$y_{imp}(t) = C\int_0^t e^{A(t-\tau)}\,B\,\delta(\tau)\,\mathrm{d}\tau = \boxed{Ce^{At}B}$$

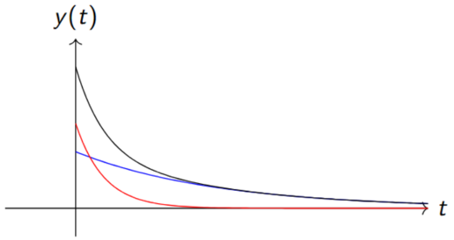
Remember how for the initial condition response, every eigenvalue of the matrix A was found in an exponential term and helped us to construct a solution?

Here, our time response is given by this exact form and matches with the result we got via laplace transform!

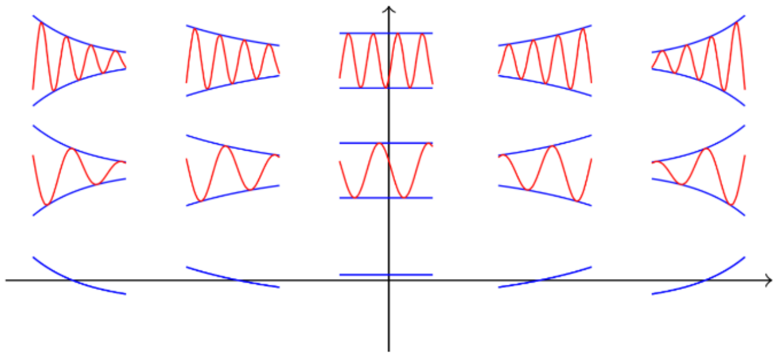
Why though??

Example

$$y(t) = Ce^{At}x_0 = \begin{bmatrix} c_1 & c_2 \end{bmatrix}\begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix}\begin{bmatrix} x_{0,1} \\ x_{0,2} \end{bmatrix} = c_1 e^{\lambda_1 t} x_{0,1} + c_2 e^{\lambda_2 t} x_{0,2}$$



Effect of Poles



Remember the derivation for the TF?

$$G(s) = C(sI - A)^{-1}B + D = \frac{N(s)}{D(s)}$$

Now considering that for a general matrix $\boldsymbol{A}$, $A^{-1} = \dfrac{\mathrm{adj}(A)}{\det(A)}$

$$\boxed{G(s) = \frac{1}{\det(sI - A)} \underbrace{C\ \mathrm{adj}(sI - A)\ B + D}_{N(s)} = \frac{N(s)}{\det(sI - A)}}$$

So now wonder that the roots of the TF are the same as the eigenvalues of matrix A

They are therefore also important for stability assessment!


Laplace?
No Laplace?

Effect of Zeros

Effect of Zeros

Lets start by considering a system where the following input - output relation is given:

$$y(t) = \frac{\mathrm{d}}{\mathrm{d}t} u(t)$$

As usual we apply our fundamental input $u(t) = e^{st}$ which will lead us to:

$$y(t) = \frac{\mathrm{d}}{\mathrm{d}t} e^{st} = s e^{st}$$

So, this tells us, that a differentiation relation is given by the TF:

$$\boxed{G_{\text{diff}}(s) = s}$$

Lets keep this in mind and consider an example!

Effect of Zeros

Consider the TF $\tilde{G}(s)$ with an added zero:

$$G(s) = \left(\frac{s}{-z} + 1\right)\tilde{G}(s) = \tilde{G}(s) + \frac{s}{-z}\tilde{G}(s)$$

When now going back to the time - domain response, remember the s means taking a derivative. Therefore:

$$\boxed{y(t) = \tilde{y}(t) + \frac{1}{-z}\dot{\tilde{y}}(t)}$$

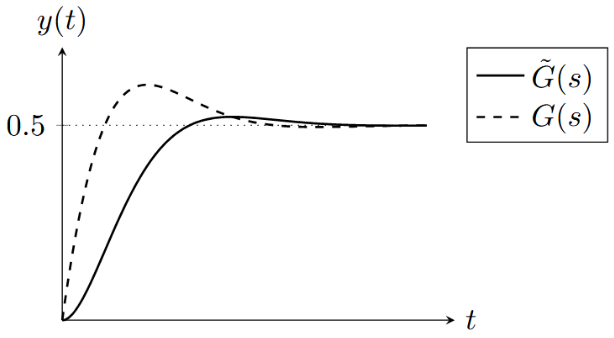
This means that we usually have an **anticipatory** effect. However, we will discuss to cases

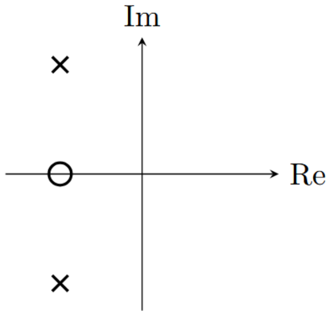
Effect of Zeros

$$\boxed{y(t) = \tilde{y}(t) + \frac{1}{-z}\dot{\tilde{y}}(t)}$$

Minimum Phase Zero

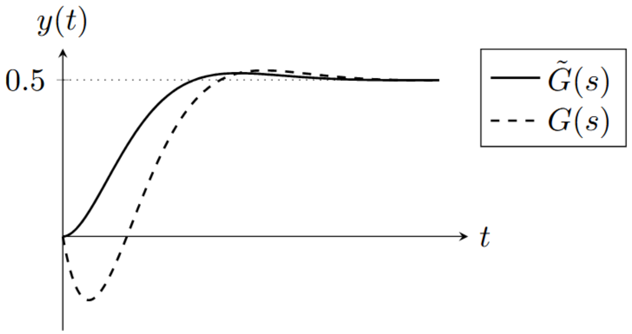
When the real part of the zero lies in the left **half plane**, it is called minimum phase. This adds a positive derivative action to the output.

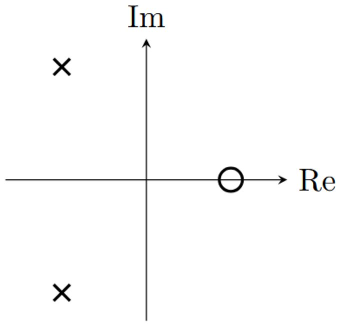




Non Minimum Phase Zero

When the real part of the zero lies in the right **half plane**, it is called non minimum phase. This adds a negative derivative action to the output (not good for controls)



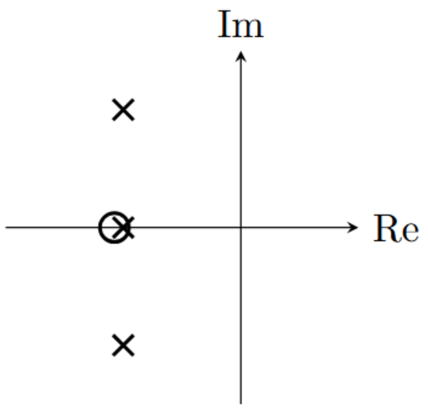


Pole - Zero Cancellation

Pole - Zero Cancellation

We would like to analyse what happens when a pole and a zero are close to each other, and or if we would cancel them in the TF. Consider the following TF;

$$G(s) = \frac{s+1+\varepsilon}{(s+1)(s+1+j)(s+1-j)}$$



Remember that looking at the partial fraction expansion, we actually see how much a pole is present in the response. So let's compute the residues:

$$r_1 = \lim_{s\to -1}(s+1)G(-1) = \varepsilon$$

$$r_2 = \lim_{s\to -1-j}(s+1+j)G(-1-j) = \frac{j-\varepsilon}{2}$$

$$r_3 = \lim_{s\to -1+j}(s+1-j)G(-1+j) = \frac{-j-\varepsilon}{2}$$

$$\boxed{G(s) = \frac{\varepsilon}{s+1} + \frac{0.5(j-\varepsilon)}{s+1+j} - \frac{0.5(j+\varepsilon)}{s+1-j}}$$

Pole - Zero Cancellation

$$\boxed{G(s) = \frac{\varepsilon}{s+1} + \frac{0.5(j-\varepsilon)}{s+1+j} - \frac{0.5(j+\varepsilon)}{s+1-j}}$$

Remember that we chose the zero being arbitrarily close to the pole. We can see now that when we **move the zero closer to the pole**, the residue of this pole**, it's effect on the response, gets smaller**, until it gets 0 eventually.

This is fine as long as we do not cancel any unstable poles. **DO NOT CANCEL UNSTABLE POLES!** (System blows up)

Initial and Final Value Theorem

Initial and Final Value Theorem

We often want to know what the steady state output of the system is. But being in the s-domain doesn't make it very interpretable.Therefore we introduce a theorem to **link the transfer function to the steady state output (s-domain to t-domain).**

Final Value Theorem

$$\boxed{\lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s)}$$

Initial Value Theorem

$$\boxed{\lim_{t \to 0} f(t) = \lim_{s \to \infty} sF(s)}$$

Application Final Value Theorem

$$\boxed{\lim_{t \to \infty} y(t) = \lim_{s \to 0} sY(s) = \lim_{s \to 0} sG(s)U(s)}$$

Final Value Theorem Proof (No black magic)

1. General true expression for Laplace Transform $\mathcal{L}\{f'(t)\} = sF(s) - f(0).$

2. Rearrange and reformulate $sF(s) = \mathcal{L}\{f'(t)\} + f(0) = \int_0^\infty e^{-st} f'(t)\, dt + f(0).$

3. Take limes $\lim_{s\to 0} sF(s) = \lim_{s\to 0} \int_0^\infty e^{-st} f'(t)\, dt + f(0) = \int_0^\infty \lim_{s\to 0} e^{-st} f'(t)\, dt + f(0) = \underbrace{\int_0^\infty f'(t)\, dt}_{} + f(0).$

4. $\lim_{s\to 0} sF(s) = [f(\infty) - f(0)] + f(0) = f(\infty).$

$\boxed{\int_0^\infty f'(t)\, dt = f(\infty) - f(0)}$

5. Reformulate

$$\boxed{\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)}$$

Example

$$\boxed{G(s) = \frac{0.2}{s^2 + s + 2}}$$

Calculate the steady state respone for G(s) when applying a dirac impulse as an input $u(t) = \delta(t)$

Example Solution

$$G(s) = \frac{0.2}{s^2 + s + 2}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Calculate the steady state respone for G(s) when applying a dirac impulse as an input $u(t) = \delta(t)$

Solving for the roots in the denominator we gat as poles: $\frac{-1 \pm j\sqrt{7}}{2}$ $Re(p_i) < 0 \Rightarrow$ Asympt. Stable

Laplace Transform of Dirac Impulse: $U(s) = 1$

In the Laplace Domain and final value theorem:

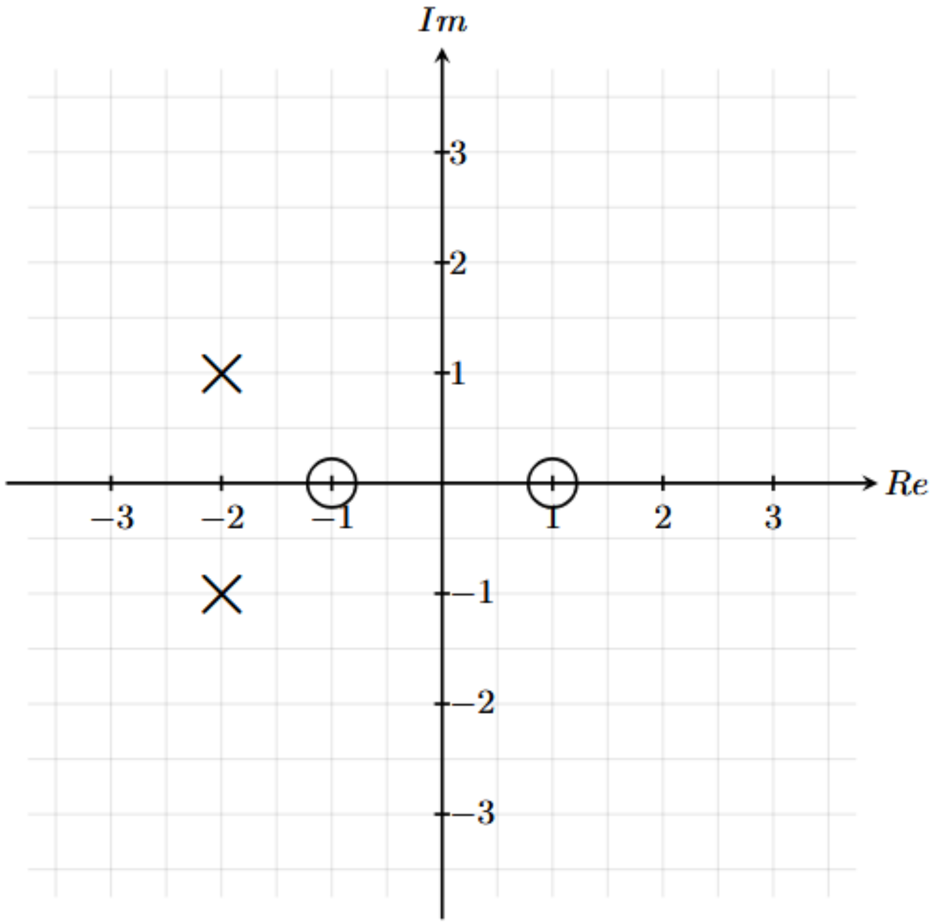
$$Y(s) = G(s)U(s) = G(s)$$

$$\lim_{t \to \infty} y(t) = \lim_{s \to 0} s\,Y(s) = \lim_{s \to 0} s\,G(s).$$

Compute the limit and response:

$$\lim_{t \to \infty} y(t) = \lim_{s \to 0} \frac{0.2\,s}{s^2 + s + 2} = 0.$$

HS 2023



What is the phase $\angle G(j)$ of $G(s = j)$?

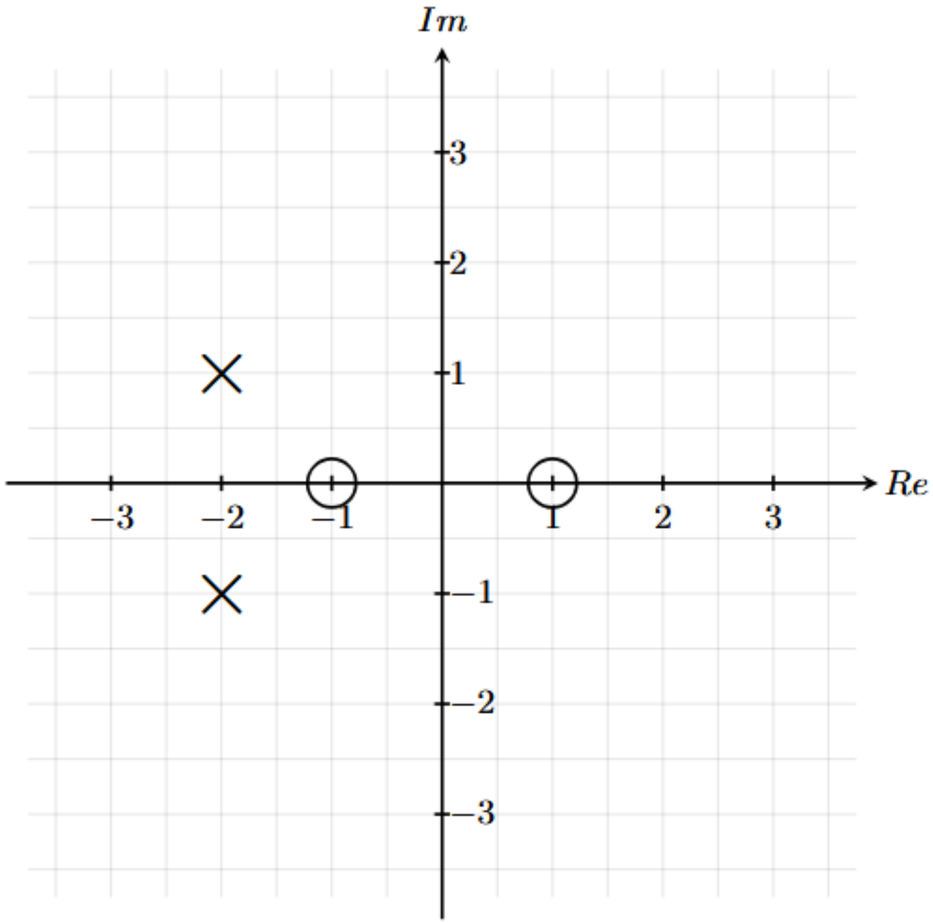
A) $= 45°$

B) $= 0°$

C) $= -45°$

D) $= 135°$

HS 2023



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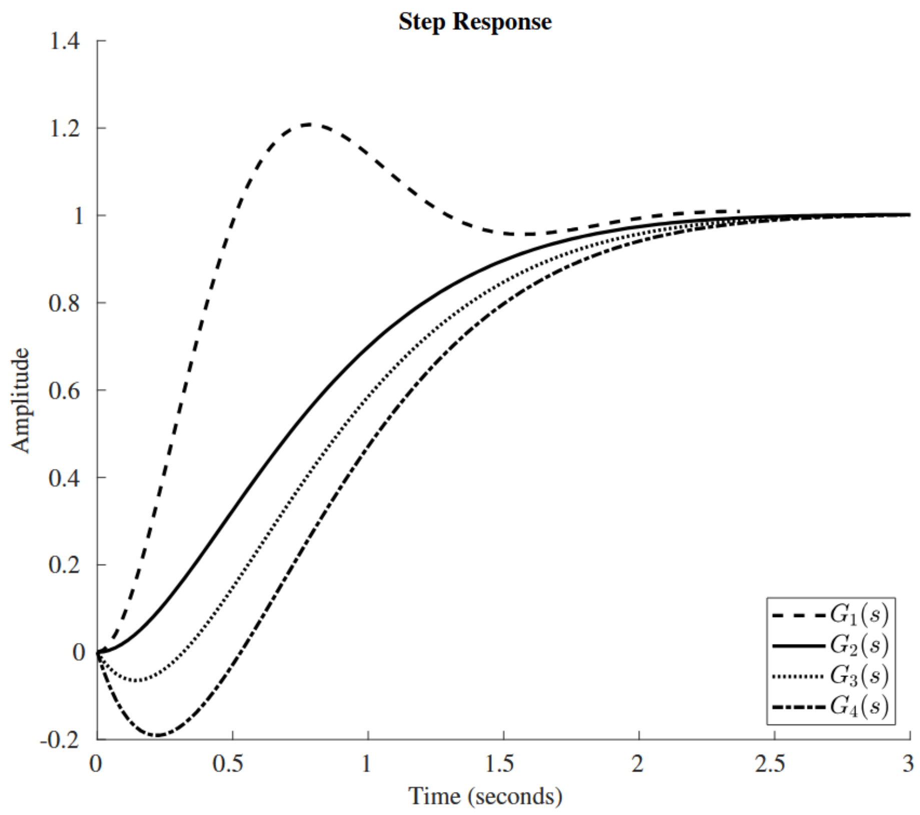
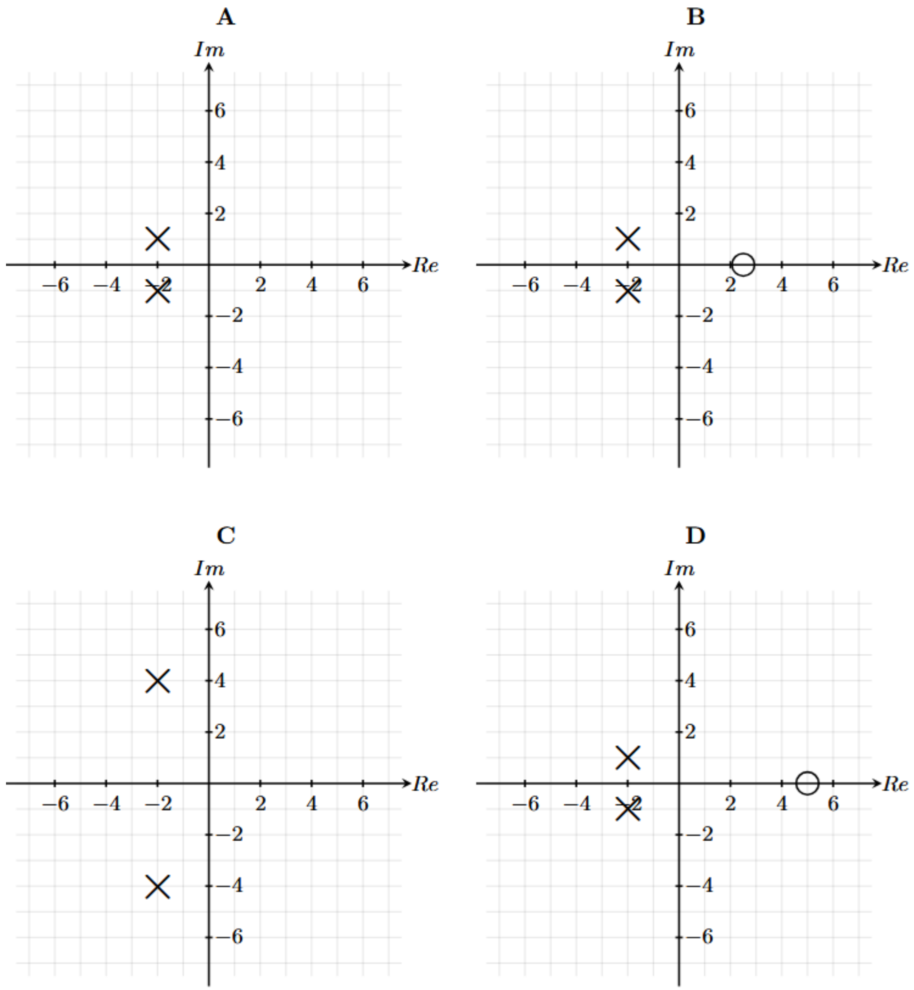




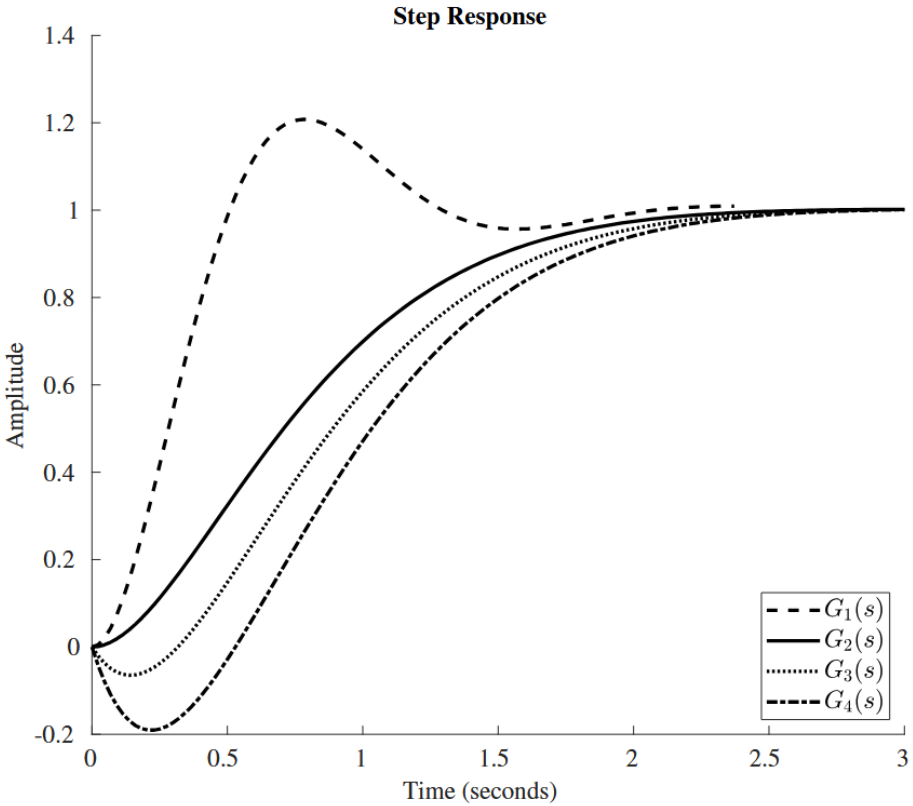
A) (A, G_1), (B, G_4), (C, G_2), (D, G_3)

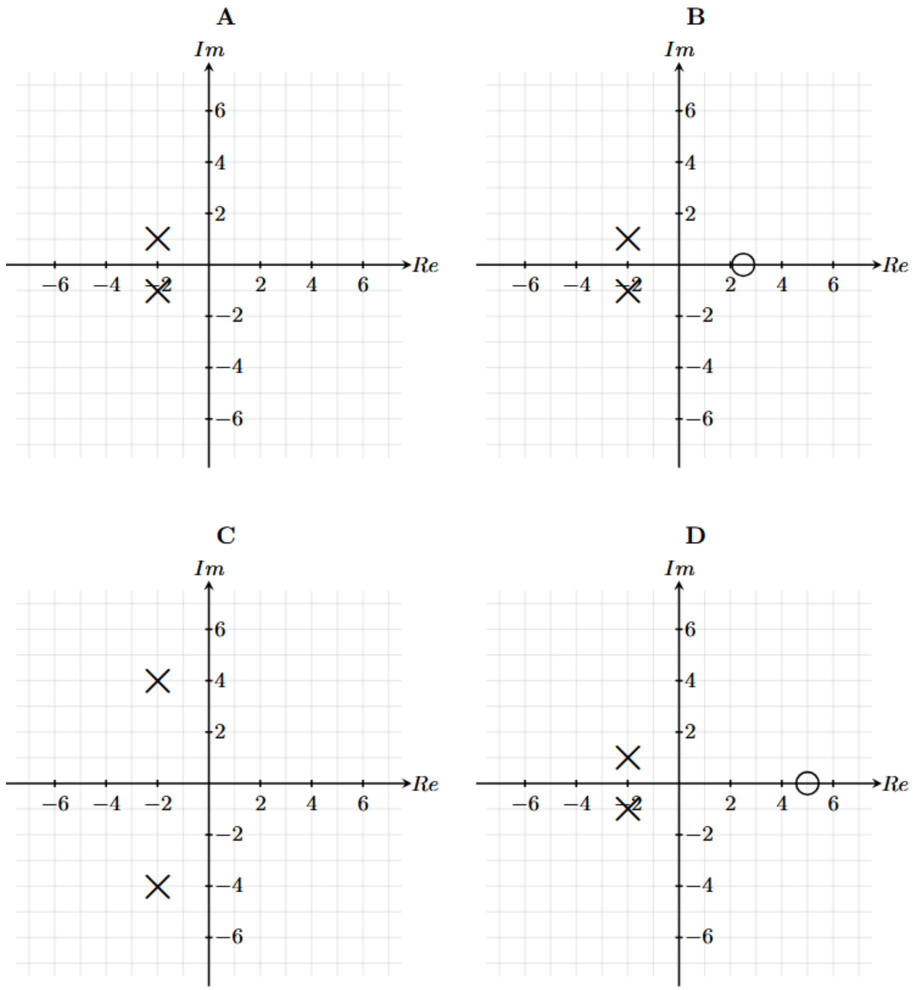
C) (A, G_3), (B, G_2), (C, G_4), (D, G_1)

B) (A, G_2), (B, G_3), (C, G_1), (D, G_4)

D) (A, G_2), (B, G_4), (C, G_1), (D, G_3)

HS 2023





A) (A, G_1), (B, G_4), (C, G_2), (D, G_3)

C) (A, G_3), (B, G_2), (C, G_4), (D, G_1)

B) (A, G_2), (B, G_3), (C, G_1), (D, G_4)

D) (A, G_2), (B, G_4), (C, G_1), (D, G_3)

HS 2023

Q22 **(1 Points)** Mark all correct statements.

- [A] In order to assess the contribution of different poles to the time response of a linear time-invariant system G it is useful to look at the transfer function of G in partial fraction expansion.
- [B] The step response of a linear time-invariant system is the same as its initial condition response with initial condition $x(0) = B$.
- [C] Performing a pole-zero cancellation in the transfer function of a linear time-invariant system does not pose threats in terms of misleading (internal) stability assessments, if and only if, the zero that is cancelled is a minimum phase zero.
- [D] For linear time-invariant systems, zeros can be interpreted as adding derivative action to the output with anticipatory effect.

A C

B D

HS 2023

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A C

B D

Q&A Session / Done

Feedback

jschultev.github.io/personal_website/Feedback